RESEARCH REPORT

PERMANENT WAVE STRUCTURES AND RESONANT TRIADS IN A LAYERED FLUID

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No. 11 February 1978

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A closed three layer fluid with small density differences between the layers has two closely related modes of gravity wave propagation. The nonlinear interactions between the wave modes are investigated, particularly the nearly resonant or significant interactions. Permanent wave solutions are calculated, and it is shown that a permanent wave of the slower mode can generate resonantly a wave harmonic of the faster mode. The equations governing resonant triads of the two modes are derived, and solutions having a permanent structure are calculated from them. It is found that some resonant triad solutions vanish when the triad is embedded in the set of all harmonics with wavenumbers in its neighbourhood.

1. INTRODUCTION

A fluid consisting of uniform layers of constant densities has a number of discrete modes of gravity wave propagation. A wave mode is associated in a loose sense with each interface and with the upper surface if open. Nonlinear interactions occur between the harmonics of the wave modes, the interactions being dominated by those that are resonant or nearly resonant. When the significant nonlinear interactions are in balance, permanent waves or permanent wave structures containing one or more wave modes occur.

The theory of permanent waves and wave structures on an open two layer fluid has been described previously [1]. The two wave modes present in that case are a fast free surface mode and a much slower interface mode. Significant interactions occur between the modes when the group velocity of a fast high wavenumber free surface harmonic matches the phase velocity of a slow low wavenumber interface harmonic. The permanent wave structure that results consists of a wave group of permanent envelope on the free surface coupled to a wave of permanent shape on the interface. There exist also single mode permanent waves on the interface or on the free surface.

The present investigation is concerned with permanent waves and wave structures inside a layered fluid resulting from significant nonlinear interactions between interface wave modes only. Observational and experimental evidence ([2], Ch.8) points to the existence of layering in the oceans and elsewhere, and by analogy with the known asymptotic properties of water waves ([3], Ch.17), permanent waves or wave structures may form an important part of the asymptotic state of wave systems generated in layered fluids. The model examined is that of a three layer closed fluid with the three layers having constant densities of similar magnitudes. The two modes of gravity wave propagation have close dispersion relations, the only major difference between the modes being that the faster mode displaces the two interfaces in phase while the slower mode displaces them in antiphase.

The wave propagation is taken to be unidirectional and spatially periodic, and the waves on the two interfaces are represented by Fourier series whose coefficients have a slow time dependence. Equations are found for the time evolution of the Fourier amplitudes in terms of all significant quadratic interactions between all Fourier amplitudes. Solutions describing permanent waves and wave structures are calculated from these equations.

Resonant triads containing harmonics from both wave modes are found to occur. One form of triad is described by

$$\omega_2(\ell) = \omega_1(k+\ell) - \omega_1(k) \tag{1.1}$$

where ω_1, ω_2 are the frequencies of the fast and slow wave modes respectively. The wavenumbers k, ℓ were such that $\ell \ll k$ in the previous investigation [1], but here k and ℓ are of comparable magnitude over much of the range of interest. This nonlinear interaction will be shown to lead to a permanent wave structure consisting of a carrier wave of the slow mode and a wave group of the fast mode.

Another form of resonant triad is described by

$$\omega_2(k) + \omega_2(\ell) = \omega_1(k+\ell). \qquad (1.2)$$

This triad did not occur in the range of parameters of the previous investigation. An important example of this triad is when, at a particular wavenumber k, the slow wave harmonic interacts resonantly with itself to generate the fast wave harmonic of wavenumber 2k, that is,

$$2\omega_2(k) = \omega_1(2k).$$

A single mode slow permanent wave cannot be generated by itself at this particular wavenumber, because its fundamental interacts with itself to generate an associated fast wave harmonic of wavenumber 2k.

Bretherton [4] showed that there is a cyclic interchange of energy between the three harmonics composing a resonant triad. The particular solution in which the energy of each of the three harmonics is constant is an elementary form of permanent wave structure. The full permanent wave structure, if it exists, can be calculated from the resonant triad by embedding it in the set of harmonics with wavenumbers in its neighbourhood.

The resonant interactions described by equations (1.1), (1.2) occur for triads of wavenumbers over much of the range of interest. This means that, if a range of harmonics of one wave mode is present, then harmonics of the other wave mode are also present in general as a result of significant quadratic interactions. Also, since the dispersion relations of the two wave modes lie close to another, some quadratic interactions may contribute significantly to the simultaneous evolution of both wave modes. This was a difficulty that did not arise in the previous investigation, where the dispersion relations were well separated and it was always possible to obtain separate evolution equations for the two wave modes.

2. GOVERNING EQUATIONS

The three layer fluid consists of layers of densities $\rho_1 > \rho_2 > \rho_3$ (from the bottom upwards) such that the density differences between the layers are small compared with the densities themselves. The total depth is h, and the depths of the three layers in nondimensional multiples of h are h_1, h_2, h_3 respectively. The fundamental wavelength is $2\pi\ell$, and a is a measure of wave amplitude. The origin is on the horizontal base, and x,y are nondimensional multiples of h, with time t a nondimensional multiple of $(h/g)^{\frac{1}{2}}$. The lower interface displacement $\eta(x,t)$ and upper interface displacement $\xi(x,t)$ are both nondimensional multiples of a, and the velocity potentials ϕ_1, ϕ_2, ϕ_3 are nondimensional multiples of $(gh)^{\frac{1}{2}}a$. The principal small parameter is $\varepsilon = a/h \ll 1$, and $\mu = h/\ell$.

The governing equations are then

$$\phi_{1xx} + \phi_{1yy} = 0 \quad , \quad 0 < y < h_1 ,$$

$$\phi_{2xx} + \phi_{2yy} = 0 \quad , \quad h_1 < y < h_1 + h_2 ,$$

$$\phi_{3xx} + \phi_{3yy} = 0 \quad , \quad h_1 + h_2 < y < h_1 + h_2 + h_3 = 1,$$

$$\phi_{1y} = 0 \quad \text{on} \quad y = 0,$$

$$\phi_{1y} - \eta_t - \varepsilon(\eta\phi_{1x})_x = 0(\varepsilon^2) \text{ on } y = h_1 ,$$

$$\phi_{2y} - \eta_t - \varepsilon(\eta\phi_{2x})_x = 0(\varepsilon^2) \text{ on } y = h_1 , \qquad (2.1)$$

$$\rho_1\phi_{1t} - \rho_2\phi_{2t} + (\rho_1 - \rho_2)\eta + \varepsilon\rho_1\eta\phi_{1yt} - \varepsilon\rho_2\eta\phi_{2yt}$$
$$+ \tfrac{1}{2}\varepsilon\rho_1(\phi_{1x}^2 + \phi_{1y}^2) - \tfrac{1}{2}\varepsilon\rho_2(\phi_{2x}^2 + \phi_{2y}^2) = 0(\varepsilon^2) \text{ on } y = h_1 ,$$

$$\phi_{2y} - \xi_t - \varepsilon(\xi\phi_{2x})_x = 0(\varepsilon^2) \text{ on } y = h_1 + h_2 ,$$

$$\phi_{3y} - \xi_t - (\xi\phi_{3x})_x = O(\varepsilon^2) \text{ on } y = h_1 + h_2,$$

$$\rho_2\phi_{2t} - \rho_3\phi_{3t} + (\rho_2 - \rho_3)\xi + \varepsilon\rho_2\xi\phi_{2yt} - \varepsilon\rho_3\xi\phi_{3yt}$$

$$+ \tfrac{1}{2}\varepsilon\rho_2(\phi_{2x}^2 + \phi_{2y}^2) - \tfrac{1}{2}\varepsilon\rho_3(\phi_{3x}^2 + \phi_{3y}^2) = O(\varepsilon^2) \text{ on } y = h_1 + h_2,$$

$$\phi_{3y} = 0 \quad \text{on} \quad y = 1.$$

Spatially periodic solutions are sought of the form

$$\xi = \frac{1}{2}\sum_{\alpha=1}^{2}\sum_{k=1}^{\infty} A_\alpha(k) \exp i\{\mu kx - \omega_\alpha(k)t\} + * ,$$

$$\eta = \frac{1}{2}\sum_{\alpha=1}^{2}\sum_{k=1}^{\infty} B_\alpha(k) \exp i\{\mu kx - \omega_\alpha(k)t\} + * ,$$

$$\phi_1 = \frac{1}{2}\sum_{\alpha=1}^{2}\sum_{k=1}^{\infty} C_\alpha(k) \cosh \mu ky \exp i\{\mu kx - \omega_\alpha(k)t\} + * , \tag{2.2}$$

$$\phi_2 = \frac{1}{2}\sum_{\alpha=1}^{2}\sum_{k=1}^{\infty} (D_\alpha(k) \cosh \mu ky + E_\alpha(k) \sinh \mu ky)$$

$$\exp i\{\mu kx - \omega_\alpha(k)t\} + * ,$$

$$\phi_3 = \frac{1}{2}\sum_{\alpha=1}^{2}\sum_{k=1}^{\infty} F_\alpha(k) \cosh \mu k(1-y) \exp i\{\mu kx - \omega_\alpha(k)t\} + * ,$$

where * denotes complex conjugate, and k, $\omega_\alpha(k)$ are nondimensional multiples of $1/\ell$, $(g/h)^{\frac{1}{2}}$ respectively. The summation with respect to α is taken over the faster wave mode with frequency $\omega_1(k)$ and the slower wave mode with frequency $\omega_2(k)$.

The two wave modes are independent in the linear approximation, when it is found that

$$r_\alpha(k) = \frac{B_\alpha(k)}{A_\alpha(k)} = \cosh \mu kh_2 + \left(\frac{\rho_3}{\rho_2 \tanh \mu kh_3} - \frac{\rho_2 - \rho_3}{\rho_2}\frac{k\mu}{\omega_\alpha^2(k)}\right)$$

$$\times \sinh \mu kh_2, \tag{2.3a}$$

$$r_1(k)\, r_2(k) = -\frac{\rho_2 - \rho_3}{\rho_1 - \rho_2}, \tag{2.3b}$$

and $\omega_1(k)$, $\omega_2(k)$ are the larger and smaller positive roots of

$$(\rho_2\rho_3T_1 + \rho_3\rho_1T_2 + \rho_1\rho_2T_3 + \rho_2^2 T_1T_2T_3)\left(\frac{\omega^2(k)}{k\mu}\right)^2$$

$$-[\rho_1(\rho_2-\rho_3)T_2T_3 + \rho_2(\rho_1-\rho_3)T_3T_1 + \rho_3(\rho_1-\rho_2)T_1T_2]\left(\frac{\omega^2(k)}{k\mu}\right)$$

$$+ (\rho_1-\rho_2)(\rho_2-\rho_3)T_1T_2T_3 = 0 \ , \qquad (2.3c)$$

where $T_j = \tanh \mu k h_j$, $j = 1,2,3$.

The equations governing the quadratic approximation are obtained by substituting the linear solution into the quadratic terms of equations (2.1). Lengthy manipulation involving the elimination of the velocity potential amplitudes $C_\alpha(k)$, $D_\alpha(k)$, $E_\alpha(k)$, $F_\alpha(k)$ leads to two differential equations (with $D = d/dt$),

$$\sum_{\alpha=1}^{2} D^2 \left[\left\{B_\alpha(k) - \left(\cosh \mu kh_2 + \frac{\rho_3}{\rho_2}\frac{\sinh \mu kh_2}{\tanh \mu kh_3}\right) A_\alpha(k)\right\}\right.$$

$$\left.\times \exp - i\omega_\alpha(k)t\right] - \sum_{\alpha=1}^{2} \frac{\rho_2-\rho_3}{\rho_2} \mu k \sinh \mu kh_2\, A_\alpha(k) \exp - i\omega_\alpha(k)t$$

$$= \frac{1}{2}\varepsilon \sum_{\beta=1}^{2}\sum_{\gamma=1}^{2}\sum_{\ell=1}^{k-1} U_{\beta\gamma}(k,-\ell)B_\beta(\ell)B_\gamma(k-\ell) \exp -i\{\omega_\beta(\ell)+\omega_\gamma(k-\ell)\}t$$

$$+ \varepsilon \sum_{\beta=1}^{2}\sum_{\gamma=1}^{2}\sum_{\ell=1}^{\infty} U_{\beta\gamma}(k,\ell)B^*_\beta(\ell)B_\gamma(k+\ell) \exp -i\{-\omega_\beta(\ell)+\omega_\gamma(k+\ell)\}t$$

$$+ O(\varepsilon^2), \quad k = 1,2,\ldots \ , \qquad (2.4a)$$

$$\sum_{\alpha=1}^{2} D^2 \left[\left\{\frac{\rho_1}{\rho_2 \tanh \mu kh_1} B_\alpha(k)\right.\right.$$

$$\left.\left. + \left(\sinh \mu kh_2 + \frac{\rho_3}{\rho_2}\frac{\cosh \mu kh_2}{\tanh \mu kh_3}\right) A_\alpha(k)\right\} \exp - i\omega_\alpha(k)t\right]$$

$$+ \sum_{\alpha=1}^{2}\left\{\frac{\rho_1-\rho_2}{\rho_2}\mu k B_\alpha(k) + \frac{\rho_2-\rho_3}{\rho_2}\mu k \cosh \mu kh_2\, A_\alpha(k)\right\}\exp -i\omega_\alpha(k)t$$

$$= \frac{1}{2}\varepsilon \sum_{\beta=1}^{2}\sum_{\gamma=1}^{2}\sum_{\ell=1}^{k-1} V_{\beta\gamma}(k,-\ell)B_\beta(\ell)B_\gamma(k-\ell)\exp -i\{\omega_\beta(\ell)+\omega_\gamma(k-\ell)\}t$$

$$+ \varepsilon \sum_{\beta=1}^{2}\sum_{\gamma=1}^{2}\sum_{\ell=1}^{\infty} V_{\beta\gamma}(k,\ell)B^*_\beta(\ell)B_\gamma(k+\ell)\exp -i\{-\omega_\beta(\ell)+\omega_\gamma(k+\ell)\}t$$

$$+ O(\varepsilon^2), \ k = 1,2,\ldots \ . \qquad (2.4b)$$

The coefficients U, V are given in the Appendix. The frequencies $\omega_\alpha(\ell)$ defined for positive ℓ by equation (2.3c) are continued to negative ℓ by $\omega_\alpha(-\ell) = -\omega_\alpha(\ell)$, $\ell > 0$.

The governing equations for any given wavenumber k cannot be reduced further when there are quadratic interactions which contribute significantly to both the faster and slower wave modes. If the quadratic interactions can be divided between those that contribute significantly only to one mode from those that contribute significantly only to the other, equations (2.4) may be divided into separate equations for the separate evolution of the faster and slower wave modes. These equations, after further manipulation, may be written

$$(D-i\omega_\alpha(k))(D+i\omega_\alpha(k))[\{B_\alpha(k) - \tilde{r}_\alpha(k)A_\alpha(k)\} \exp-i\omega_\alpha(k)t]$$

$$= \frac{1}{2}\varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{k-1} P_{\beta\gamma}(k,-\ell)B_\beta(\ell)B_\gamma(k-\ell) \exp-i\{\omega_\beta(\ell)+\omega_\gamma(k-\ell)\}t$$

$$+ \varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{\infty} P_{\beta\gamma}(k,\ell)B^*_\beta(\ell)B_\gamma(k+\ell) \exp-i\{-\omega_\beta(\ell)+\omega_\gamma(k+\ell)\}t$$

$$+ O(\varepsilon^2), \quad \alpha = 1,2; \; k = 1,2,\ldots \quad ,(2.5a)$$

[The notation $\tilde{\omega}_\alpha(k)$, $\tilde{r}_\alpha(k)$ introduced here refers to the variable for the other mode, that is,

$\tilde{\omega}_\alpha(k) = \omega_1(k)\omega_2(k)/\omega_\alpha(k)$, $\tilde{r}_\alpha(k) = r_1(k)r_2(k)/r_\alpha(k)$.]

$$(D-i\tilde{\omega}_\alpha(k))(D+i\tilde{\omega}_\alpha(k))[\{B_\alpha(k)-r_\alpha(k)A_\alpha(k)\} \exp-i\omega_\alpha(k)t]$$

$$= \frac{1}{2}\varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{k-1} Q_{\beta\gamma}(k,-\ell)B_\beta(\ell)B_\gamma(k-\ell) \exp-i\{\omega_\beta(\ell)+\omega_\gamma(k-\ell)\}t$$

$$+ \varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{\infty} Q_{\beta\gamma}(k,\ell)B^*_\beta(\ell)B_\gamma(k+\ell) \exp-i\{-\omega_\beta(\ell)+\omega_\gamma(k+\ell)\}t$$

$$+ O(\varepsilon^2), \quad \alpha = 1,2; \; k = 1,2,\ldots \quad .(2.5b)$$

The β,γ summation in both sets of equations is taken over all contributing significant quadratic interactions, and the coefficients P,Q are given in the Appendix.

The significant quadratic interactions are those which are resonant or lie near resonance. More precisely, they occur with triads for which either

$$\begin{aligned} &\omega_\alpha(k) - \omega_\beta(\ell) - \omega_\gamma(k-\ell) = 0(\varepsilon) \\ \text{or}\quad &\omega_\alpha(k) + \omega_\beta(\ell) - \omega_\gamma(k+\ell) = 0(\varepsilon) \end{aligned} \tag{2.6a}$$

Equations (2.5) are valid only when such quadratic interactions do not contribute significantly to both modes, that is, when

$$\begin{aligned} &\tilde{\omega}_\alpha(k) - \omega_\beta(\ell) - \omega_\gamma(k-\ell) = 0(1), \\ \text{and}\quad &\tilde{\omega}_\alpha(k) + \omega_\beta(\ell) - \omega_\gamma(k+\ell) = 0(1). \end{aligned} \tag{2.6b}$$

Equations (2.5b) may then be integrated twice, to yield

$$\begin{aligned} &\{B_\alpha(k) - r_\alpha(k)A_\alpha(k)\} \exp - i\omega_\alpha(k)t \\ &= \tfrac{1}{2}\varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{k-1} \frac{Q_{\beta\gamma}(k,-\ell)}{\tilde{\omega}_\alpha^2(k) - \{\omega_\beta(\ell)+\omega_\gamma(k-\ell)\}^2} B_\beta(\ell)B_\gamma(k-\ell) \\ &\qquad \times \exp - i\{\omega_\beta(\ell)+\omega_\gamma(k-\ell)\}t \\ &+ \varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{\infty} \frac{Q_{\beta\gamma}(k,\ell)}{\tilde{\omega}_\alpha^2(k) - \{-\omega_\beta(\ell)+\omega_\gamma(k+\ell)\}^2} B_\beta^*(\ell)B_\gamma(k+\ell) \\ &\qquad \times \exp - i\{-\omega_\beta(\ell)+\omega_\gamma(k+\ell)\}t + 0(\varepsilon^2), \quad \alpha = 1,2; \; k = 1,2,.. \end{aligned} \tag{2.7a}$$

Equation (2.5a) may be integrated once only, to give

$$\begin{aligned} &(D + i\omega_\alpha(k))[\{B_\alpha(k) - \tilde{r}_\alpha(k)A_\alpha(k)\} \exp - i\omega_\alpha(k)t] \\ &= \tfrac{1}{2}i\varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{k-1} \frac{P_{\beta\gamma}(k,-\ell)}{\omega_\alpha(k)+\omega_\beta(\ell)+\omega_\gamma(k-\ell)} B_\beta(\ell)B_\gamma(k-\ell) \\ &\qquad \exp - i\{\omega_\beta(\ell)+\omega_\gamma(k-\ell)\}t \\ &+ i\varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{\infty} \frac{P_{\beta\gamma}(k,\ell)}{\omega_\alpha(k)-\omega_\beta(\ell)+\omega_\gamma(k+\ell)} B_\beta^*(\ell)B_\gamma(k+\ell) \\ &\qquad \exp - i\{-\omega_\beta(\ell)+\omega_\gamma(k+\ell)\}t + 0(\varepsilon^2), \quad \alpha = 1,2; \; k = 1,2,.. \end{aligned} \tag{2.7b}$$

Further elimination leads to evolution equations for each Fourier amplitude, namely

$$DA_\alpha(k) = \frac{1}{2}i\varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{k-1} R_{\alpha\beta\gamma}(k,-\ell)B_\beta(\ell)B_\gamma(k-\ell)$$

$$\times \exp i\{\omega_\alpha(k) - \omega_\beta(\ell)-\omega_\gamma(k-\ell)\}t$$

$$+ i\varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{\infty} R_{\alpha\beta\gamma}(k,\ell)B^*_\beta(\ell)B_\gamma(k+\ell) \exp i\{\omega_\alpha(k)+\omega_\beta(\ell)-\omega_\gamma(k+\ell)\}t$$

$$+ O(\varepsilon^2), \quad \alpha = 1,2; \; k = 1,2,\ldots \quad , \qquad (2.8a)$$

$$DB_\alpha(k) = \frac{1}{2}i\varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{k-1} S_{\alpha\beta\gamma}(k,-\ell)B_\beta(\ell)B_\gamma(k-\ell)$$

$$\times \exp i\{\omega_\alpha(k)-\omega_\beta(\ell)-\omega_\gamma(k-\ell)\}t$$

$$+ i\varepsilon \sum_{\beta,\gamma} \sum_{\ell=1}^{\infty} S_{\alpha\beta\gamma}(k,\ell)B^*_\beta(\ell)B_\gamma(k+\ell) \exp i\{\omega_\alpha(k)+\omega_\beta(\ell)-\omega_\gamma(k+\ell)\}t$$

$$+ O(\varepsilon^2), \quad \alpha = 1,2; \; k = 1,2,\ldots \quad , \qquad (2.8b)$$

The β,γ summation in equations (2.7) and (2.8) is taken over all quadratic interactions which contribute significantly in the sense of equations (2.6), and the coefficients R,S are stated in terms of P,Q in the Appendix.

The dispersion relations for the two wave modes may be deduced from equation (2.3c). In the small wavenumber limit, the two long wave velocities (as multiples of $(gh)^{\frac{1}{2}}$) are the two positive roots of

$$(\rho_1\rho_2h_3 + \rho_2\rho_3h_1 + \rho_3\rho_1h_2)c_0^4 - \{\rho_1(\rho_2-\rho_3)h_2h_3$$

$$+ \rho_2(\rho_1-\rho_3)h_1h_3 + \rho_3(\rho_1-\rho_2)h_1h_2\}c_0^2 + (\rho_1-\rho_2)(\rho_2-\rho_3)h_1h_2h_3 = 0. \qquad (2.9)$$

Equation (2.3c) may be solved in the large wavenumber limit to give

$$\frac{\omega^2(k)}{\mu k} = \frac{\rho_2 - \rho_3}{\rho_2 + \rho_3} \quad \text{or} \quad \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \qquad (2.10)$$

Hence the dispersion relation for each wave mode behaves linearly near the origin according to $\omega(k) \sim \mu k c_0$ for each c_0, and tends asymptotically towards the parabolic shape of equation (2.10) as k becomes large.

The parameters used in subsequent calculations are $\rho_1 = 1.05$, $\rho_2 = 1$, $\rho_3 = 0.95$, $h_1 = 0.4$, $h_2 = 0.2$, $h_3 = 0.4$. The long wave velocities are then 0.142, 0.063, and the asymptotic dispersion relations for large k are $\omega_1(k) \sim 0.160(k\mu)^{\frac{1}{2}}$ and $\omega_2(k) \sim 0.156(k\mu)^{\frac{1}{2}}$. The phase velocities and group velocities of the two wave modes are drawn in figure 1.

3. RESONANT TRIADS

A triad of harmonics whose frequencies satisfy

$$\omega_\alpha(k) + \omega_\beta(\ell) - \omega_\gamma(k+\ell) = O(\varepsilon)$$

has evolution equations, from equations (2.8), given by

$$DB_\alpha(k) = i\varepsilon S_{\alpha\beta\gamma}(k,\ell)B_\beta^*(\ell)B_\gamma(k+\ell)\exp i\{\omega_\alpha(k)+\omega_\beta(\ell)-\omega_\gamma(k+\ell)\}t, \tag{3.1a}$$

$$DB_\beta(\ell) = i\varepsilon S_{\beta\alpha\gamma}(\ell,k)B_\alpha^*(k)B_\gamma(k+\ell)\exp i\{\omega_\alpha(k)+\omega_\beta(\ell)-\omega_\gamma(k+\ell)\}t, \tag{3.1b}$$

$$DB_\gamma(k+\ell) = i\varepsilon S_{\gamma\beta\alpha}(k+\ell,-\ell)B_\alpha(k)B_\beta(\ell)\exp -i\{\omega_\alpha(k)+\omega_\beta(\ell)-\omega_\gamma(k+\ell)\}t, \tag{3.1c}$$

except that when $\alpha = \beta$ and $k = \ell$ the quadratic term of equation (3.1c) is divided by 2. This is the set of equations solved by Bretherton [4], §6, in a scaled form.

The particular solutions of interest here are those corresponding to an elementary permanent wave structure, with

$$\eta = b_\alpha \cos\{\mu k(x-ct)\} + b_\beta \cos\{\mu\ell(x-ct) - nt\} + b_\gamma \cos\{\mu(k+\ell)(x-ct) - nt\},$$

$$\xi = (b_\alpha/r_\alpha(k))\cos\{\mu k(x-ct)\} + (b_\beta/r_\beta(\ell))\cos\{\mu\ell(x-ct) - nt\} + (b_\gamma/r_\gamma(k+\ell))\cos\{\mu(k+\ell)(x-ct) - nt\},$$

where the latter equation has a validity subject to equations (2.7a). The amplitudes b_α, b_β, b_γ are constants, c is the wave velocity of the structure, and n is the frequency of the group relative to the structure. The corresponding Fourier amplitudes are then

$$B_\alpha(k) = b_\alpha \exp i\{\omega_\alpha(k) - \mu kc\}t, \tag{3.2a}$$

$$B_\beta(\ell) = b_\beta \exp i\{\omega_\beta(\ell) - \mu\ell c - n\}t, \tag{3.2b}$$

$$B_\gamma(k+\ell) = b_\gamma \exp i\{\omega_\gamma(k+\ell) - \mu(k+\ell)c - n\}t, \tag{3.2c}$$

and equations (3.1) become

$$\{\omega_\alpha(k) - \mu kc\}b_\alpha = \varepsilon S_{\alpha\beta\gamma}(k,\ell)b_\beta b_\gamma, \tag{3.3a}$$

$$\{\omega_\beta(\ell) - \mu\ell c - n\}b_\beta = \varepsilon S_{\beta\alpha\gamma}(\ell,k)b_\alpha b_\gamma, \tag{3.3b}$$

$$\{\omega_\gamma(k+\ell) - \mu(k+\ell)c - n\}b_\gamma = \varepsilon S_{\gamma\beta\alpha}(k+\ell,-\ell)b_\alpha b_\beta, \tag{3.3c}$$

except that when $\alpha = \beta$ and $k = \ell$ the quadratic term of equation (3.3c) is divided by 2. The amplitude of the carrier wave (b_α) is given, and the amplitude of the group envelope may be specified independently, making 5 equations for the 5 parameters b_α, b_β, b_γ, c, n.

The first of the resonant triads (equation 1.1) is rewritten now (with the μ-dependence explicit)

$$\omega_2(\mu) = \omega_1\{(k+\tfrac{1}{2})\mu\} - \omega_1\{(k-\tfrac{1}{2})\mu\}, \tag{3.4}$$

which for $k \gg 1$ is equivalent to

$$c_2(\mu) = c_{g1}(k\mu). \tag{3.5}$$

If a line is drawn across figure 1 in the direction of the $k\mu$-axis, the intercept with the c_2-curve defines μ and the intercept with the c_{g1}-curve defines $k\mu$ in equation (3.5). The solution of equation (3.4) for k in terms of μ is drawn in figure 2. It can be seen by comparing figures 1 and 2 that equation (3.5) is a reasonable approximation to equation (3.4) even for values of k near 1. The lowest resonant triad is that for which $k = 3/2$, at $\mu = 2.06$, when $c_2(2.06) = 0.062$ and $c_{g1}(3.09) = 0.060$.

Solutions of the resonant triad equations (3.3) for this and for other values of μ have been calculated, and the triads then embedded in the range of harmonics with wave-numbers in the neighbourhood of resonance. It was found that as the number of interacting harmonics was increased, the solutions always converged towards permanent wave structures consisting of a carrier wave from the slower wave mode and a group of waves from the faster wave mode. A typical calculation is described in §5.

The second of the resonant triads (equation 1.2) is rewritten now (with the μ-dependence explicit)

$$\omega_2(\mu) + \omega_2(k\mu) = \omega_1\{(k+1)\mu\}, \qquad (3.6)$$

and its solution for k in terms of μ is drawn in figure 3. The lowest triad is that for which $k = 1$ at $\mu = 4.50$, and the minimum value of μ is 0.433 when $k = 57$. Elementary permanent wave structure solutions of equations (3.3) have been calculated for this resonant triad for a number of values of μ. When the triad was embedded in the range of harmonics with wavenumbers in its neighbourhood, the solution vanished for each of the values of μ calculated. The nonlinear interactions with the neighbouring harmonics negated the balance between the harmonics in the original triad. An example of such a calculation is described in §5.

A permanent wave consisting of the fast mode alone has the form

$$\xi = \sum_{k=1}^{\infty} a(k) \cos \mu k(x-ct), \tag{4.1a}$$

$$\eta = \sum_{k=1}^{\infty} r_1(k)a(k) \cos \mu k(x-ct), \tag{4.1b}$$

where $r_1(k)$ is positive and is given by equation (2.3a). The total displacement of the upper interface is 2a, where a is the measure of amplitude introduced in §2, so that

$$\sum_{k=1}^{\infty} a(2k-1) = 1, \tag{4.2}$$

from the definition of ε. The corresponding Fourier amplitudes are

$$A_1(k) = a(k) \exp i\{\omega_1(k) - \mu kc\}t,$$

$$B_1(k) = r_1(k)a(k) \exp i\{\omega_1(k) - \mu kc\}t, \quad k = 1,2,\ldots,$$

which on substitution into equations (2.8a) yield

$$\{\omega_1(k) - \mu kc\}a(k) = \frac{1}{2}\varepsilon \sum_{\ell=1}^{k-1} R_{111}(k,-\ell)r_1(\ell)r_1(k-\ell)a(\ell)a(k-\ell)$$

$$+ \varepsilon \sum_{\ell=1}^{\infty} R_{111}(k,\ell)r_1(\ell)r_1(k+\ell)a(\ell)a(k+\ell), \quad k = 1,2,\ldots \tag{4.3}$$

These equations may be solved numerically by the method outlined previously ([1], §3). The two interfaces are displaced in phase by the fast permanent wave. In the limit as $\mu \to 0$, the fast permanent wave tends towards a solitary wave of elevation of both interfaces.

Although a permanent wave consisting of the slower mode alone exists for most values of μ, at some values of μ it exists only in association with a harmonic of the faster wave mode. If a fast harmonic is not present, a permanent wave has the form

$$\eta = \sum_{k=1}^{\infty} b(k) \cos \mu k(x-ct), \tag{4.4a}$$

$$\xi = \sum_{k=1}^{\infty} (b(k)/r_2(k)) \cos \mu k(x-ct), \tag{4.4b}$$

where $r_2(k)$ is negative and is given by equation (2.3a). The origin is measured from a trough of the wave on the lower interface, so that

$$\sum_{k=1}^{\infty} b(2k-1) = -1. \tag{4.5}$$

The wave velocity c for small values of ε is approximated by the wave velocity of the fundamental, that is,

$$\mu c = \omega_2(\mu),$$

(with the μ-dependence of ω_2 stated explicitly). The frequency of the kth harmonic of the permanent wave is then $k\omega_2(\mu)$, and resonant forcing of the fast wave harmonic of the same wavenumber occurs if

$$k\omega_2(\mu) = \omega_1(k\mu). \tag{4.6}$$

The solution of this equation for k in terms of μ is drawn in figure 4. For each integer value of k, the figure shows the value of μ at which a slow permanent wave forces resonantly a fast wave harmonic of this particular wavenumber, beginning with k = 2 when μ = 4.50.

The representation of the slow permanent wave in equations (4.4) must therefore be generalised to include a fast wave harmonic at certain values of μ, that is

$$\eta = \sum_{k=1}^{\infty} \{b(k) + r_1(k)a(k)\} \cos \mu k(x-ct), \tag{4.7a}$$

$$\xi = \sum_{k=1}^{\infty} \{b(k)/r_2(k) + a(k)\} \cos \mu k(x-ct), \tag{4.7b}$$

where $a(k)$ is $O(\varepsilon)$ for most k and μ, but is $O(1)$ in the neighbourhood of the curve of resonance in figure 4. The corresponding Fourier amplitudes are then

$$A_1(k) = a(k) \exp i\{\omega_1(k) - \mu kc\}t, \tag{4.8a}$$

$$A_2(k) = (b(k)/r_2(k)) \exp i\{\omega_2(k) - \mu kc\}t, \tag{4.8b}$$

$$B_1(k) = r_1(k)a(k) \exp i\{\omega_1(k) - \mu kc\}t, \tag{4.8c}$$

$$B_2(k) = b(k) \exp i\{\omega_2(k) - \mu kc\}t, \tag{4.8d}$$

$k = 1,2,\ldots,$ where the validity of equations (4.8b), (4.8c) is subject to equations (2.7a). The governing equations are now equations (2.4) because there are some quadratic interactions which contribute significantly to both wave modes. When the Fourier amplitudes from equations (4.8) are substituted into equations (2.4) and combined with equation (4.5), a set of algebraic equations for $a(k)$, $b(k)$, $k = 1,2,\ldots,$ and c is obtained which may be solved numerically by the method used for the fast permanent waves.

The first point of resonance is at $\mu = 4.50$, where the fast second harmonic of amplitude $a(2)$ is generated resonantly. Solutions for the slow permanent wave in the neighbourhood of $\mu = 4.50$ include a significant contribution from $a(2)$ with a negligible contribution from all other $a(k)$. The contribution from a(2) as a function of μ is drawn in figure 5.

A simple oscillator forced in the neighbourhood of resonance is described by the differential equation

$$(D^2 + n^2)x = f \cos mt$$

with the forced solution

$$x = \frac{f}{n^2 - m^2} \cos mt \quad , \quad m \neq n.$$

Its amplitude $f/(n^2 - m^2)$, as a function of the forcing frequency m, has an asymptote at right angles to the m-axis at the natural frequency n, and contains a phase change of π as m crosses n. The amplitude of the forced harmonic a(2) in figure 5 displays a similar behaviour as a function of the fundamental wavenumber μ, except that the asymptote is not quite at right angles to the μ-axis, and hence the solutions for a(2) have a small overlap near resonance. Although the fast second harmonic has a similar behaviour to the simple oscillator described above, it differs in that the central interaction is modified by the neighbouring weaker interactions taking place with the other harmonics present. The consequence is that for any value of μ in the immediate neighbourhood of resonance, a slow permanent wave has associated with it one or other of two possible fast second harmonic waves, the two possible waves differing in phase by π.

The fast third harmonic a(3) is generated resonantly for values of μ in the neighbourhood of 2.72, where it is found that the behaviour of a(3) as a function of μ is the same as in figure 5 except that the scale is reduced. The magnitudes and bandwidths of the resonantly generated higher fast harmonics are found to decrease as μ decreases further, until as $\mu \to 0$ in the solitary wave limit, the associated fast harmonics also tend towards zero. The solitary wave is one of depression on the lower interface with an associated elevation of the upper interface.

5. CALCULATIONS OF PERMANENT WAVE STRUCTURES

$\mu = 2$

The two resonant triads at $\mu = 2$ are described by (with the μ-dependence explicit)

$$\omega_2(\mu) = \omega_1(2.04\mu) - \omega_1(1.04\mu), \tag{5.1a}$$

and
$$\omega_2(\mu) + \omega_2(4.40\mu) = \omega_1(5.40\mu), \tag{5.1b}$$

with
$$3.96\ \omega_2(\mu) = \omega_1(3.96\mu). \tag{5.1c}$$

Equation (5.1a) is satisfied by a resonant triad consisting of a slow wave of wavenumber 1 and fast waves of wavenumbers 1 and 2. An elementary permanent wave structure may be formed from this resonant triad with interface displacements

$$\xi = (b(1)/r_2(1)) \cos \mu(x-ct) + a(1) \cos\{\mu(x-ct)-nt\} + a(2) \cos\{2\mu(x-ct) - nt\},$$

$$\eta = b(1) \cos \mu(x-ct) + a(1)r_1(1) \cos\{\mu(x-ct) - nt\} + a(2)r_1(2) \cos\{2\mu(x-ct) - nt\}.$$

Equations (3.3) applied to this resonant triad are

$$\{\omega_2(1) - \mu c\}b(1) = \varepsilon\ S_{211}(1,1)r_1(1)r_1(2)a(1)a(2), \tag{5.2a}$$

$$\{\omega_1(1) - \mu c - n\}a(1) = \varepsilon\ R_{121}(1,1)r_1(2)b(1)a(2), \tag{5.2b}$$

$$\{\omega_1(2) - 2\mu c - n\}a(2) = \varepsilon\ R_{112}(2,-1)r_1(1)a(1)b(1). \tag{5.2c}$$

The origin is measured from the trough on the lower interface of the slow carrier wave, so that equation (4.5) becomes

$$b(1) = -1. \tag{5.2d}$$

The amplitude of the envelope of the fast waves may be assigned independently. If this amplitude is taken to be the same as that of the carrier wave, a short calculation shows that this condition may be written either

$$a(1) - a(2) = 1 , \tag{5.2e}$$

or
$$a(1) + a(2) = 1 , \tag{5.2f}$$

depending on whether the envelope maximum is to lie over the crest or the trough on the lower interface of the carrier wave.

Two solutions of the set of equations (5.2) were found for the case $\mu = 2$. These are

(i) $b(1) = -1$, $a(1) = 0.28$, $a(2) = -0.72$, $c = 0.062$, $n = 0.124$,

and

(ii) $b(1) = -1$, $a(1) = 0.64$, $a(2) = 0.36$, $c = 0.062$, $n = 0.118$.

Each solution was then embedded in the set of all harmonics with wavenumbers in its neighbourhood, the number of harmonics being increased until there were no further changes in the two solutions to the numerical precision used (10^{-3}). When the number of harmonics of the slow carrier wave is increased to four, equation (5.1c) shows that the interactions generating the fourth harmonic also generate resonantly the fast harmonic of wavenumber four. The governing equations are therefore equations (2.4) together with the geometric constraints of equation (4.5) and the extended forms of equations (5.2e) or (5.2f). The two solutions become

(i) carrier wave: $b(1) = -0.92$, $b(2) = -0.30$, $b(3) = -0.08$,
$b(4) = -0.02$, $a(4) = 0.02$;

wave group: $a(1) = 0.34$, $a(2) = -0.65$;

$c = 0.063$, $n = 0.120$;

(ii) carrier wave: $b(1) = -0.91$, $b(2) = -0.31$, $b(3) = -0.09$,
$b(4) = -0.02$, $a(4) = 0.02$;

wave group: $a(1) = 0.59$, $a(2) = 0.44$, $a(3) = -0.02$;

$c = 0.062$, $n = 0.116$.

It can be seen that for both solutions, the resonant triad provides a good approximation because it contains the interaction which dominates the complete permanent wave structure.

The two solutions are sketched in figures 6(a), 6(b). In the first, the group envelope maximum coincides with the flat section of the carrier wave (crest on the lower interface and trough on the upper interface), while in the second the group envelope maximum coincides with the peaked section of the carrier wave (trough on the lower interface and crest on the upper interface).

$\mu = 1.2$

The two resonant triads at $\mu = 1.2$ are described by

$$\omega_2(\mu) = \omega_1(2.99\mu) - \omega_1(1.99\mu), \tag{5.3a}$$

and $$\omega_2(\mu) + \omega_2(9.93\mu) = \omega_1(10.93\mu), \tag{5.3b}$$

with $$6.44\ \omega_2(\mu) = \omega_1(6.44\mu). \tag{5.3c}$$

Equation (5.3b) is satisfied by a resonant triad consisting of slow harmonics of wavenumbers 1 and 10 and a fast harmonic of wavenumber 11. An elementary permanent wave structure is sought with interface displacements

$$\xi = (b(1)/r_2(1))\cos\{\mu(x-ct)\} + (b(10)/r_2(10))\cos\{10\mu(x-ct)-nt\} + a(11)\cos\{11\mu(x-ct)-nt\},$$

$$\eta = b(1)\cos\{\mu(x-ct)\} + b(10)\cos\{10\mu(x-ct)-nt\} + a(11)r_1(11)\cos\{11\mu(x-ct)-nt\},$$

for which equations (3.3) become

$$\{\omega_2(1) - \mu c\}b(1) = \varepsilon\, S_{221}(1,10)r_1(11)b(10)a(11), \tag{5.4a}$$

$$\{\omega_2(10) - 10\mu c - n\}b(10) = \varepsilon\, S_{221}(10,1)r_1(11)b(1)a(11), \tag{5.4b}$$

$$\{\omega_1(11) - 11\mu c - n\}a(11) = \varepsilon\, R_{122}(11,-1)b(1)b(10). \tag{5.4c}$$

Equation (5.2d) is unchanged, namely

$$b(1) = -1 , \tag{5.4d}$$

and equations (5.2e) and (5.2f) become

$$b(10)/r_2(10) - a(11) = 1 , \tag{5.4e}$$

and

$$b(10)/r_2(10) + a(11) = 1 . \tag{5.4f}$$

Two solutions were found for equations (5.4), the first satisfying equation (5.4e) and the second satisfying equation (5.4f). These are

(i) $b(1) = -1$, $b(10) = -0.48$, $a(11) = -0.64$, $c = 0.063$, $n = -0.233$,

and

(ii) $b(1) = -1$, $b(10) = -0.57$, $a(11) = 0.56$, $c = 0.063$, $n = -0.229$.

When each of these resonant triad solutions was embedded in the set of all harmonics with wavenumbers in its neighbourhood, either no solution could be obtained or an unrealistic solution was found. Further harmonics were added in different ways, but there was no suggestion of convergence towards a permanent wave structure. A spurious convergence could be obtained by eliminating interactions of the type described by equation (5.3a), which suggests that the first form of resonant triad is dominant in permanent wave structures. There was good convergence to a permanent wave structure centred on the resonant triad consisting of a slow carrier wave of wavenumber 1 and a fast group of wavenumbers 2 and 3, as described by equation (5.3a).

$\mu = 0.25$

The first resonant triad is described by

$$\omega_2(\mu) = \omega_1(12.26\mu) - \omega_1(11.26\mu) , \tag{5.5a}$$

there is no second resonant triad at this value of μ, and

$$30.5\ \omega_2(\mu) = \omega_1(30.5\mu) \ . \tag{5.5b}$$

Although equation (5.5a) is the exact solution for equation (3.4) at $\mu = 0.25$, it is nearly true for a broad waveband in k near k = 12 because of the steepness of the $k-\mu$ curve in figure 2 for small values of μ. For this reason, the permanent wave structures at this value of μ are found by a method developed previously ([1], §6), as follows.

The first step is to calculate the harmonics of the slow permanent wave at $\mu = 0.25$ together with its associated fast harmonic. These harmonics are then taken as a first approximation for the carrier wave of the permanent wave structure, so that equations (2.8) become linear algebraic equations for the harmonics of the wave group with eigenvalues equal to the frequency n of the waves in the group relative to the group envelope. The approximate wave structure solutions found by solving equations (2.8) are then used as starting estimates for Newton-Raphson solutions of equations (2.4).

The first of the solutions is sketched in figure 7. It is related to the first of the solutions at $\mu = 2$, with the group envelope maximum coinciding with the flat section of the carrier wave. In the previous investigation [1], the form of the group could be interpreted in terms of the horizontal velocity field generated by the carrier wave. A similar interpretation is more difficult here because of the complex nature of the horizontal velocity field, including the discontinuities in this field across the interfaces. The example here is one of very long wavelength, suggesting that in the long wave limit a permanent wave structure consists of a uniform sinusoidal wave train everywhere except in the neighbourhood of the solitary carrier wave, where the envelope of the wave train decreases to a minimum at the maximum displacement of the carrier wave.

6. GENERALISATION

The previous and present investigations are combined now to make some general deductions about permanent waves and wave structures in layered fluids. A fluid consisting of a number of layers of constant densities of similar magnitudes with an open upper surface has a finite set of discrete closely related slow internal wave modes and a fast free surface wave mode. Harmonics from the fast free surface mode form a fast free surface periodic permanent wave, but internal periodic permanent waves consisting principally of harmonics from one of the internal wave modes are associated at many wavelengths with a harmonic or harmonics from the other internal wave modes, through resonant interactions of the type described by equation (4.6).

The central property of permanent wave structures is that the phase velocity of the carrier wave matches the group velocity of the wave group, with the carrier wave consisting of the slower mode alone and the group consisting of the faster mode alone. A layered fluid having three or more wave modes would support permanent wave structures of this type consisting of two of the wave modes. It is suggested that it may also support permanent wave structures consisting of three or more wave modes, with the slowest mode providing the carrier wave and the group velocities of each of the faster modes matching the phase velocity of the carrier wave. This property could be tested by making calculations on a three layer open fluid to determine whether permanent wave structures exist consisting of all three wave modes.

To summarise, it is suggested that a layered fluid would support permanent waves consisting solely or predominantly of one wave mode in which all harmonics have the same phase (or a phase difference of π), and it would support also permanent wave structures in which the slowest mode present is the carrier wave of permanent shape and the other modes present are wave groups of permanent envelope.

REFERENCES

1. P.J. Bryant Permanent wave structures on an open two layer fluid. *Stud. Appl. Math.* **57**, 225-246 (1977).

2. J.S. Turner *Buoyancy effects in fluids.* C.U.P. (1973)

3. G.B. Whitham *Linear and nonlinear waves.* Wiley. (1974)

4. F.P. Bretherton Resonant interactions between waves. The case of discrete oscillations. *J. Fluid Mech.* **20**, 457-479 (1964).

APPENDIX

Denote $\omega_k = \omega_\alpha(k)$, $\omega_\ell = \omega_\beta(\ell)$, $\omega_{k\ell} = \omega_\gamma(k+\ell)$,

$$y_1 = \frac{\omega_\ell}{\tanh \mu\ell h_1} + \frac{\omega_{k\ell}}{\tanh \mu(k+\ell)h_1}$$

$$y_2 = \frac{\mu\ell}{\omega_\ell} + \frac{\mu(k+\ell)}{\omega_{k\ell}}$$

$$y_3 = \omega_\ell^2 + \omega_{k\ell}^2 - \omega_\ell\,\omega_{k\ell},$$

$$y_4 = \frac{\omega_\ell\,\omega_{k\ell}}{\tanh \mu\ell h_1 \tanh \mu(k+\ell)h_1}$$

$$y_5 = \left(\frac{\rho_1\,\omega_\ell}{\rho_2 \tanh \mu\ell h_1} - \frac{(\rho_1-\rho_2)\mu\ell}{\rho_2\,\omega_\ell}\right) \times \left(\frac{\rho_1\,\omega_{k\ell}}{\rho_2 \tanh \mu(k+\ell)h_1} - \frac{(\rho_1-\rho_2)\mu(k+\ell)}{\rho_2\,\omega_{k\ell}}\right)$$

$$y_6 = \frac{\omega_\ell}{\tanh \mu\ell h_3} + \frac{\omega_{k\ell}}{\tanh \mu(k+\ell)h_3}$$

$$y_7 = \frac{\omega_\ell\,\omega_{k\ell}}{\tanh \mu\ell h_3 \tanh \mu(k+\ell)h_3}$$

$$y_8 = \left(\frac{\rho_3\,\omega_\ell}{\rho_2 \tanh \mu\ell h_3} - \frac{(\rho_2-\rho_3)\mu\ell}{\rho_2\,\omega_\ell}\right) \times \left(\frac{\rho_3\,\omega_{k\ell}}{\rho_2 \tanh \mu(k+\ell)h_3} - \frac{(\rho_2-\rho_3)\mu(k+\ell)}{\rho_2\,\omega_{k\ell}}\right).$$

Then

$$U_{\beta\gamma}(k,\ell) = \frac{1}{2}\mu k\,(\omega_{k\ell}-\omega_\ell)\left(\frac{\rho_1-\rho_2}{\rho_2}\,y_2 - \frac{\rho_1}{\rho_2}\,y_1\right) + \frac{\mu k(\rho_2-\rho_3)\cosh\{\mu k h_2\}(\omega_{k\ell}-\omega_\ell)}{2\rho_2\,r_\beta(\ell)\,r_\gamma(k+\ell)}\,y_2 - \frac{\mu k\,\rho_3\,(\omega_{k\ell}-\omega_\ell)}{2\rho_2\,r_\beta(\ell)\,r_\gamma(k+\ell)}\left(\cosh \mu k h_2 + \frac{\sinh \mu k h_2}{\tanh \mu k h_3}\right) y_6 + \frac{\mu k \sinh \mu k h_2}{2\,r_\beta(\ell)\,r_\gamma(k+\ell)}\left(y_8 - \frac{\rho_2-\rho_3}{\rho_2}\,y_3 - \frac{\rho_3}{\rho_2}\,y_7\right),$$

$$V_{\beta\gamma}(k,\ell) = -\frac{\mu k\,\rho_1\,(\omega_{k\ell}-\omega_\ell)}{2\rho_2 \tanh \mu k h_1}\,y_1 + \frac{1}{2}\mu k\left(\frac{\rho_1-\rho_2}{\rho_2}\,y_3 - \frac{\rho_1}{\rho_2}\,y_4 + y_5\right) - \frac{\mu k(\rho_2-\rho_3)\sinh\{\mu k h_2\}(\omega_{k\ell}-\omega_\ell)}{2\rho_2\,r_\beta(\ell)\,r_\gamma(k+\ell)}\,y_2$$

$$+ \frac{\mu k\, \rho_3\, (\omega_{k\ell} - \omega_\ell)}{2\,\rho_2\, r_\beta(\ell)\, r_\gamma(k+\ell)} \left(\sinh \mu k h_2 + \frac{\cosh \mu k h_2}{\tanh \mu k h_3}\right) y_6$$

$$+ \frac{\mu k \cosh \mu k h_2}{2\, r_\beta(\ell)\, r_\gamma(k+\ell)} \left(\frac{\rho_2 - \rho_3}{\rho_2}\, y_3 - y_8 + \frac{\rho_3}{\rho_2}\, y_7\right),$$

$$P_{\beta\gamma}(k,\ell) = \left(1 - \frac{\rho_1\, \omega_k^{\,2}}{(\rho_1 - \rho_2)\,\mu k \tanh \mu k h_1}\right) U_{\beta\gamma}(k,\ell)$$

$$+ \frac{\rho_2\, \omega_k^{\,2}}{(\rho_1 - \rho_2)\,\mu k}\, V_{\beta\gamma}(k,\ell),$$

$$Q_{\beta\gamma}(k,\ell) = \left(1 - \frac{\rho_1\, \tilde{\omega}_k^{\,2}}{(\rho_1 - \rho_2)\,\mu k \tanh \mu k h_1}\right) U_{\beta\gamma}(k,\ell)$$

$$+ \frac{\rho_2\, \tilde{\omega}_k^{\,2}}{(\rho_1 - \rho_2)\,\mu k}\, V_{\beta\gamma}(k,\ell),$$

$$R_{\alpha\beta\gamma}(k,\ell) = \frac{1}{r_\alpha(k) - \tilde{r}_\alpha(k)} \left(\frac{\rho_2\, P_{\beta\gamma}(k,\ell)}{(\rho_1 - \rho_2)(\omega_k + \omega_{k\ell} - \omega_\ell)}\right.$$

$$\left. - \frac{\omega_k + \omega_\ell - \omega_{k\ell}}{\tilde{\omega}_k^{\,2} - (\omega_{k\ell} - \omega_\ell)^2}\, Q_{\beta\gamma}(k,\ell)\right),$$

$$S_{\alpha\beta\gamma}(k,\ell) = \frac{1}{r_\alpha(k) - \tilde{r}_\alpha(k)} \left(\frac{\rho_2\, r_\alpha(k)\, P_{\beta\gamma}(k,\ell)}{(\rho_1 - \rho_2)(\omega_k + \omega_{k\ell} - \omega_\ell)}\right.$$

$$\left. + \frac{\omega_k + \omega_\ell - \omega_{k\ell}}{\tilde{\omega}_k^{\,2} - (\omega_{k\ell} - \omega_\ell)^2}\, \tilde{r}_\alpha(k)\, Q_{\beta\gamma}(k,\ell)\right).$$

It is noted that the terms in $Q_{\beta\gamma}(k,\ell)$ in the equations for $R_{\alpha\beta\gamma}(k,\ell)$ and $S_{\gamma\beta\gamma}(k,\ell)$ should be omitted to be consistent with equations (2.6a). They were retained because some calculations extended over triads far from resonance as well as all triads near resonance.

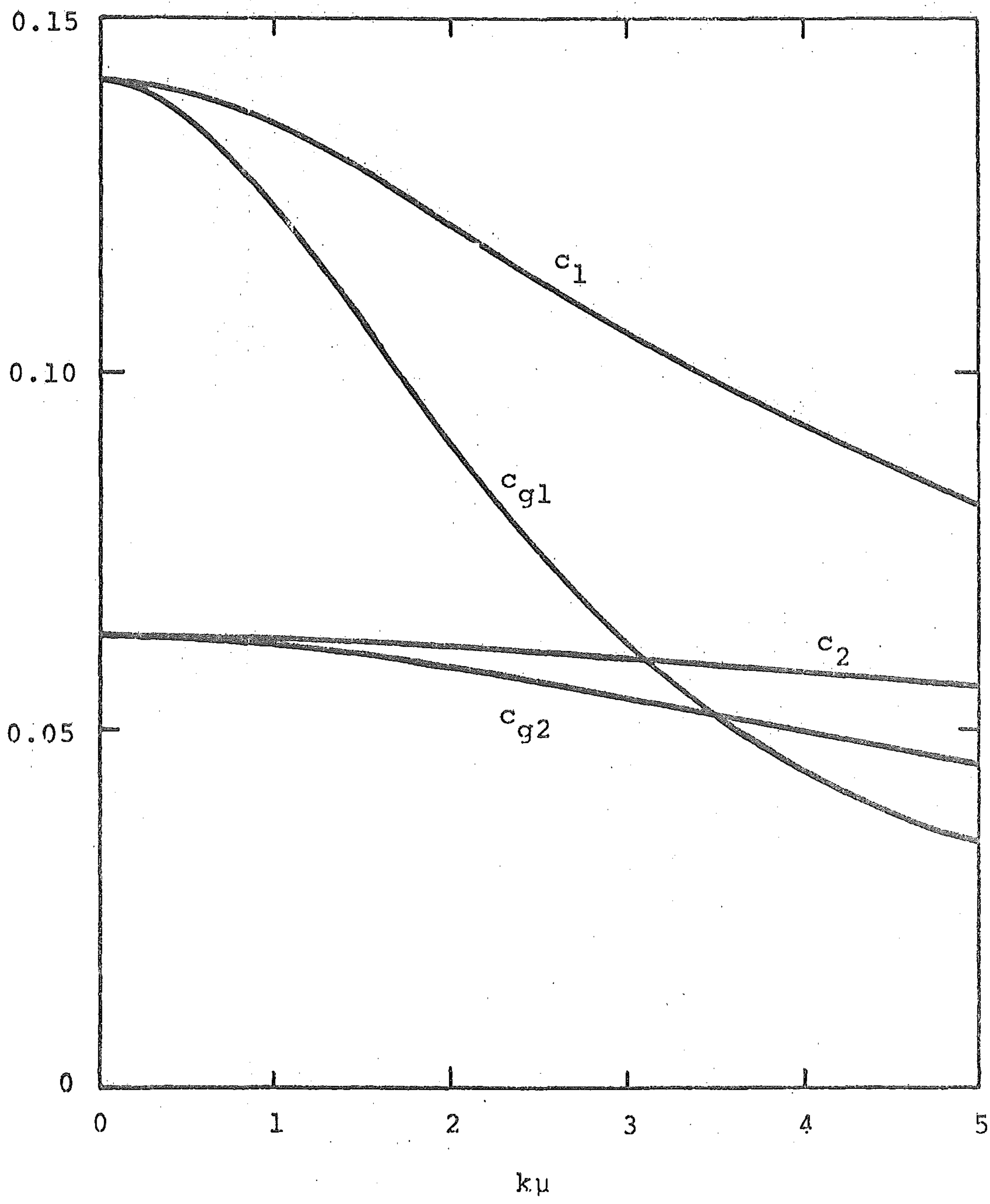

Figure 1. Phase velocity and group velocity of the two wave modes.

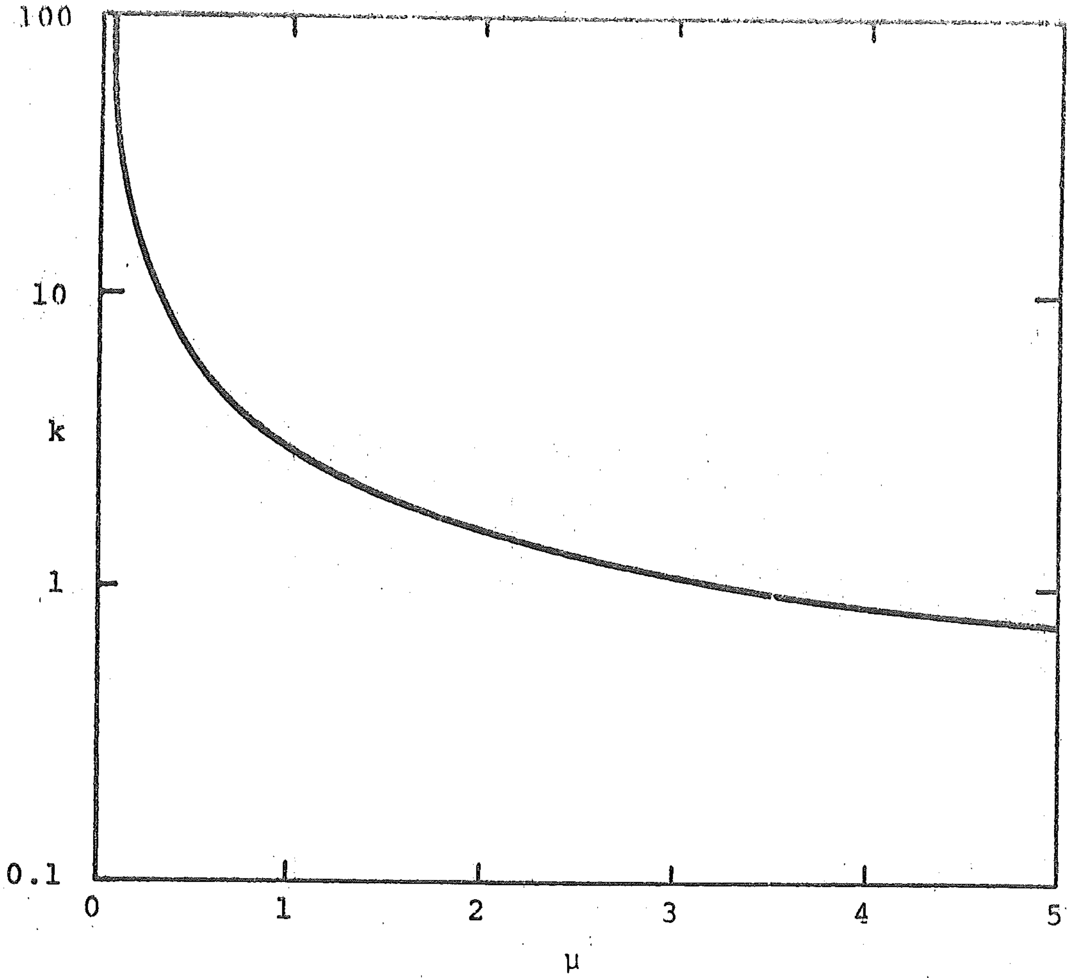

Figure 2. Solution of equation (3.4).

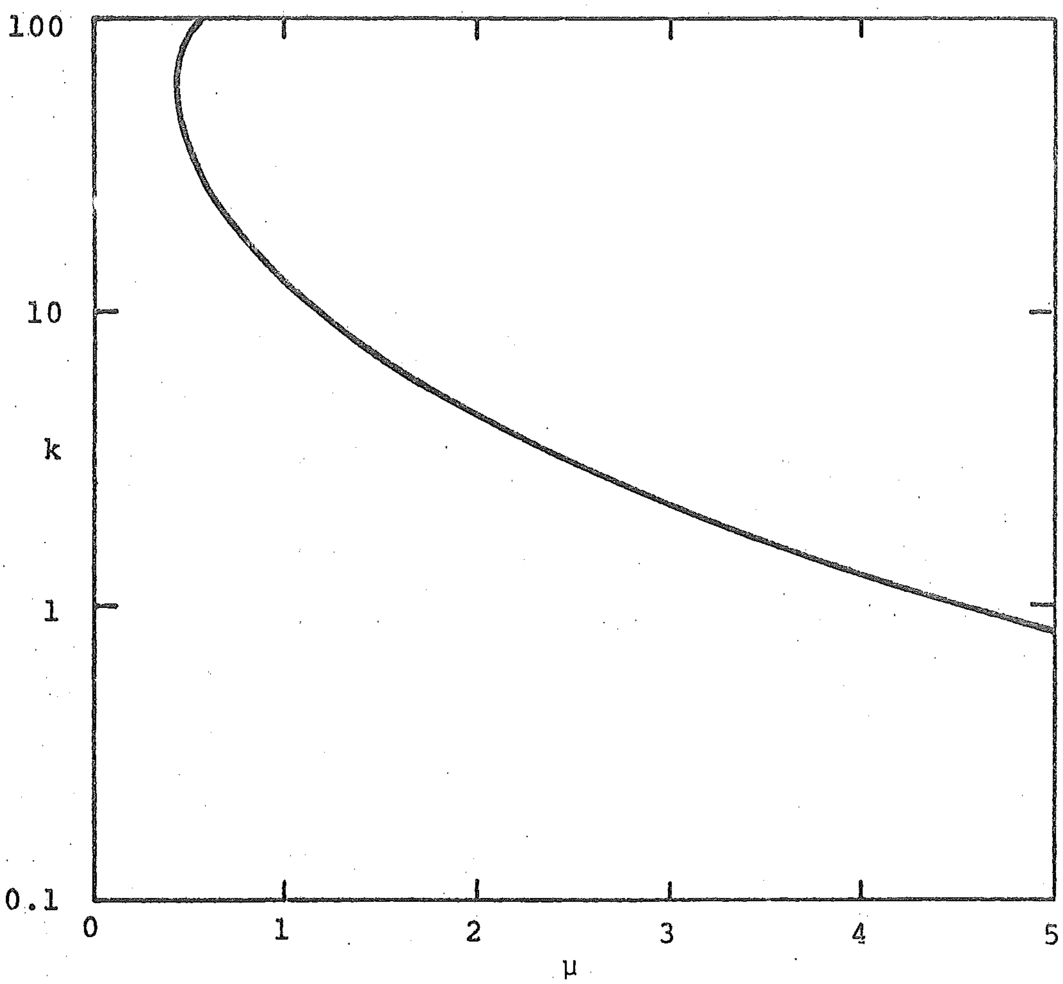

Figure 3. Solution of equation (3.6).

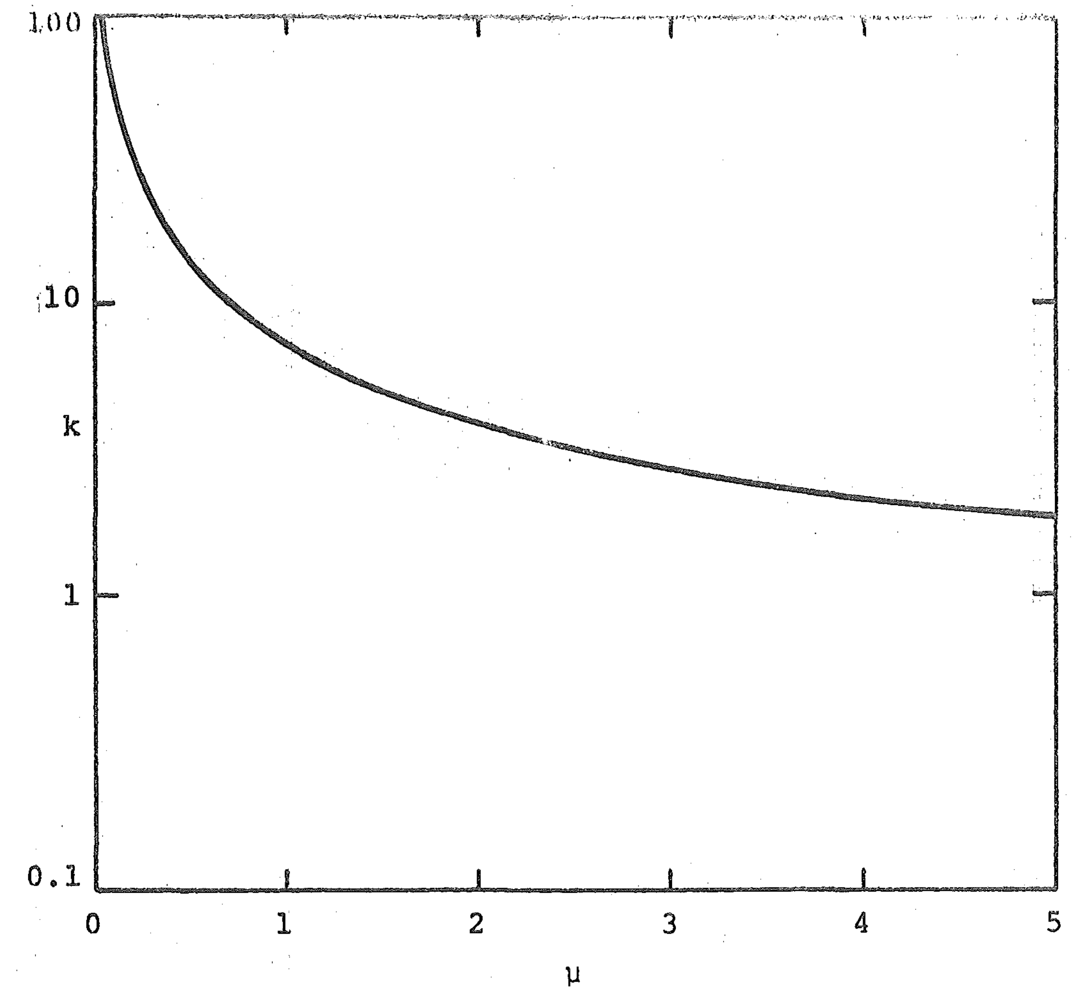

Figure 4. Solution of equation (4.6).

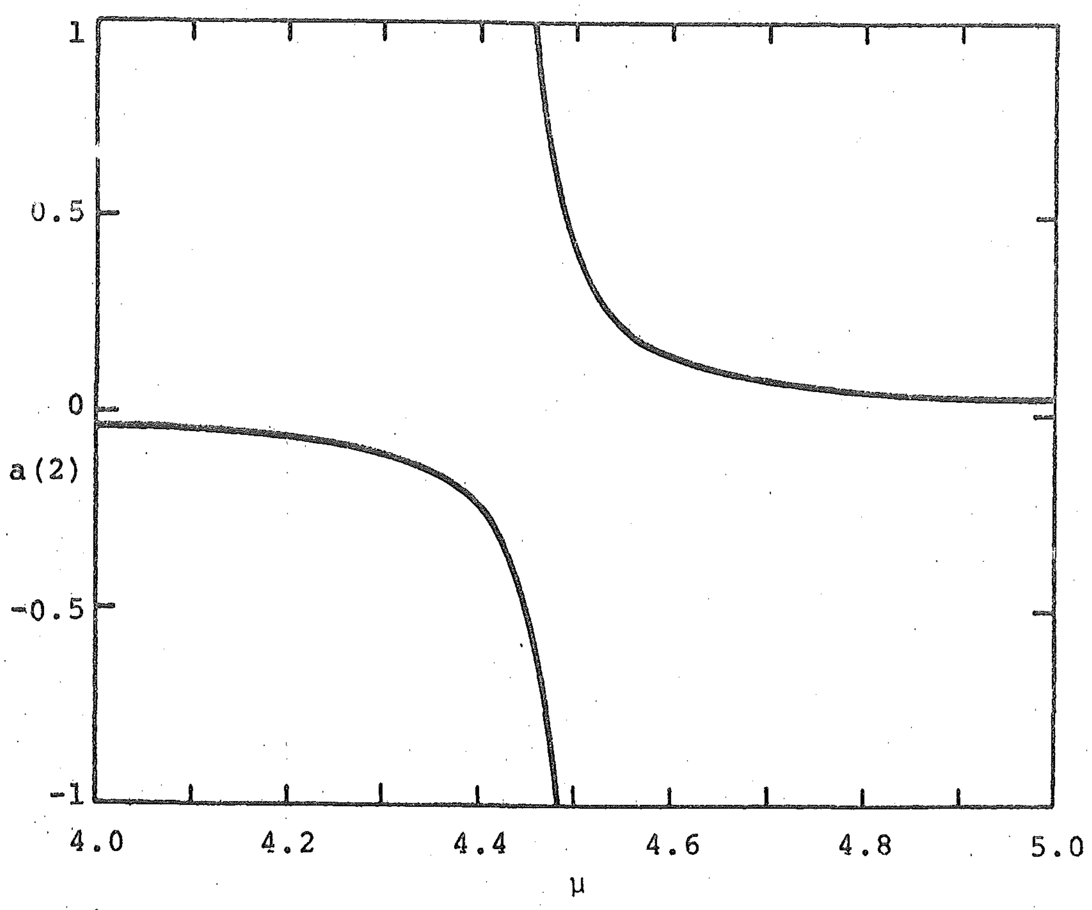

Figure 5. Resonant generation of fast second harmonic.

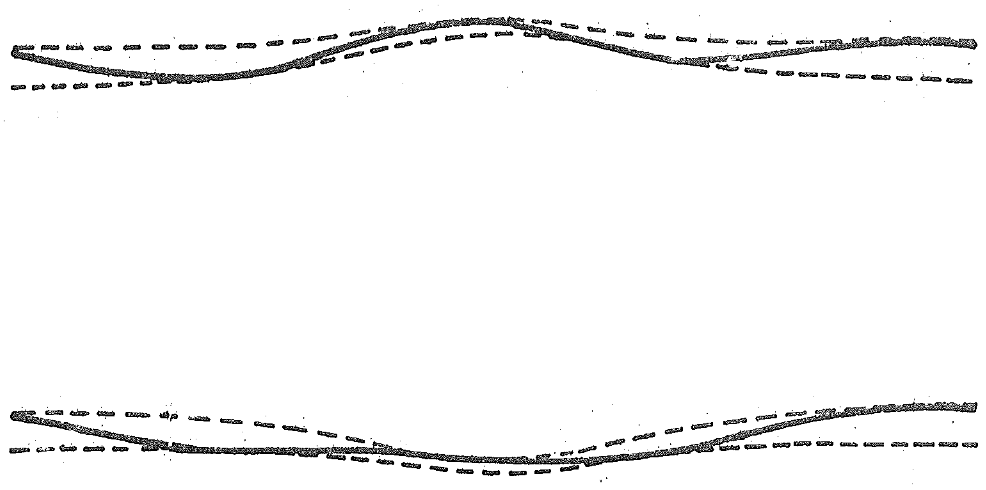

Figure 6(a). One wavelength of the first form of a permanent wave structure at $\mu = 2$ (vertical magnification 2π). The dashed curves show the envelopes of permanent form, and the solid curves the interface displacements at an instant.

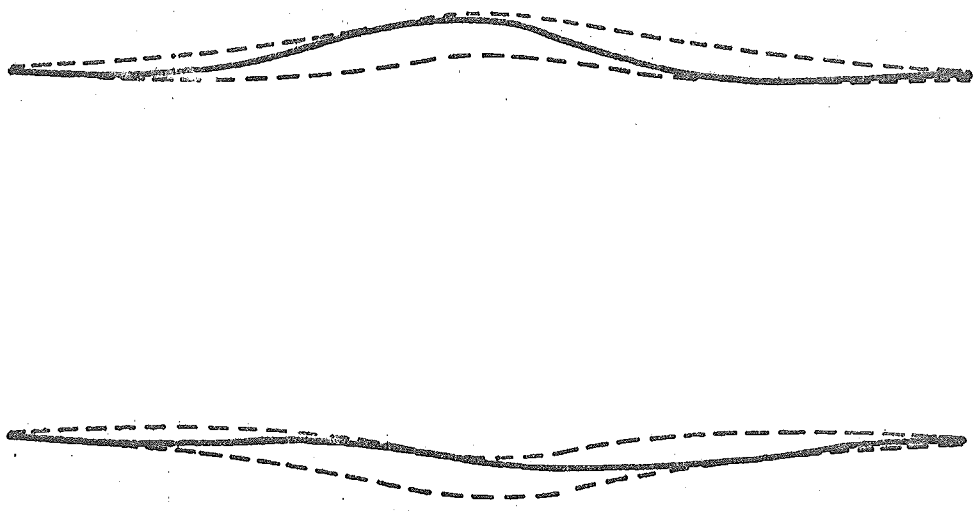

Figure 6(b). Second form of the same structure.

Figure 7. One wavelength of the first form of a permanent wave structure at $\mu = 0.25$ (vertical magnification 2π, horizontal reduction 8).

PREVIOUS MATHEMATICAL RESEARCH REPORTS

1. (November 1976) BRYANT, P.J.
   Permanent wave structures on an open two layer fluid.

2. (December 1976) WEIR, G.J.
   Conformal Killing tensors in reducible spaces.

3. (January 1977) WEIR, G.J. & KERR, R.P.
   Diverging type-D metrics.

4. (February 1977) KERR, R.P. & WILSON, W.B.
   Singularities in the Kerr-Schild metrics.

5. (May 1977) PETERSEN, G.M.
   Algebras for matrix limitation.

6. (August 1977) GRAY, A.G.
   Topological methods in second order arithmetic.

7. (September 1977) WAYLEN, P.C.
   Green functions in the early universe.

8. (November 1977) BEATSON, R.K.
   The asymptotic cost of Lagrange interpolatory side conditions in the space C(T).

9. (November 1977) BEATSON, R.K.
   The degree of monotone approximation.

10. (November 1977) BREACH, D.R.
    On the non-existence of 5-(24,12,6) and 4-(23,11,6) designs.